

# THE TWISTED KÄHLER-RICCI FLOW

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**ABSTRACT.** In this paper we study a generalization of the Kähler-Ricci flow, in which the Ricci form is twisted by a closed, non-negative  $(1,1)$ -form. We show that when a twisted Kähler-Einstein metric exists, then this twisted flow converges exponentially. This generalizes a result of Perelman on the convergence of the Kähler-Ricci flow, and it builds on work of Tian-Zhu.

## 1. INTRODUCTION

The Kähler-Ricci flow, introduced by Hamilton [12] has been studied extensively in recent years. In this paper we study a generalization of the Kähler-Ricci flow, which we call the *twisted* Kähler-Ricci flow. Fix a compact Kähler manifold  $(M, J)$ .

**Definition 1.1.** *Let  $\alpha$  be a closed, non-negative  $(1,1)$ -form on  $(M, J)$ , and suppose that  $2\pi c_1(M) - \alpha > 0$  is a Kähler class. Fix  $\omega_0 \in 2\pi c_1(M) - \alpha$ . The normalized twisted Kähler-Ricci flow is the evolution equation*

$$(1) \quad \frac{\partial}{\partial t} \omega = \omega + \alpha - \text{Ric}(\omega),$$

with  $\omega(0) = \omega_0$  at  $t = 0$ .

The normalized twisted Kähler-Ricci flow preserves the cohomology class of  $\omega$ , and the short and long-time existence results follow from the standard arguments for the Kähler-Ricci flow on a Fano manifold (see e.g. Cao [3], or the book [7]). Below we will also consider an unnormalized version of the flow. When  $\alpha = 0$ , both of these flows reduce to the usual Kähler-Ricci flow on the underlying Fano manifold  $(M, J)$ . Our main object of study in this paper is the convergence of the flow (1) when a solution of the equation

$$(2) \quad \text{Ric}(\omega) = \omega + \alpha$$

exists. Solutions of this equation are called twisted Kähler-Einstein metrics, and they arise in various settings, for instance in Fine [9] and Song-Tian [19]. Of particular interest recently has been the generalization where  $\alpha$  is a multiple of the current of integration along a divisor, in relation with Kähler-Einstein metrics which have conical singularities (see for example Donaldson [8], Jeffres-Mazzeo-Rubinstein [13]). In this paper, however, we will focus on the case when  $\alpha$  is a smooth form.

Our main result is the following.

**Theorem 1.** *Suppose that there is a solution  $\omega$  of equation (2). Then for any  $\omega_0 \in [\omega]$ , the flow (1) with initial metric  $\omega_0$  converges exponentially fast to a (perhaps different) solution of (2).*

Similar results can be proved in the case when  $c_1(M) - \alpha \leq 0$ , but this is essentially contained in the work of Cao [3]. In the case when  $\alpha = 0$ , our main theorem reduces to an unpublished result of Perelman. Namely, we obtain as a corollary;

**Corollary 1.** *Suppose the Fano manifold  $(M, J)$  admits a Kähler-Einstein metric. Then for any  $\omega_0 \in c_1(M)$ , the Kähler-Ricci flow with initial metric  $\omega_0$  converges exponentially fast to a Kähler-Einstein metric.*

This theorem has been addressed several times in the literature, most notably by Tian-Zhu and collaborators (see [26, 25, 24]). Our approach in this paper is based on the ideas in [26], in particular we make strong use of the result of Tian-Zhu [26] that Perelman's entropy functional increases to a fixed topological constant along the Kähler-Ricci flow.

The first step in the proof of Theorem 1 is an extension of Perelman's estimates [17] to the twisted flow. For our later applications, we require uniform control of the constants appearing Perelman's estimates for a family of twisted Kähler-Ricci flows with initial metrics lying in a bounded family in  $C^3$ . This requires us to reformulate the arguments in [17] in order to obtain effective bounds. The presence of the extra form  $\alpha$  causes little difficulty, although at various points it is important that  $\alpha$  is closed and non-negative. In addition, we extend to the twisted case the uniform Sobolev inequality along the Kähler-Ricci flow, proved by Zhang [28]. These developments appear in Sections 2 and 3. With these results established, we show in Section 4 that Perelman's entropy functional increases along the twisted KRF to a fixed topological constant, extending a result of Tian-Zhu [26] to the twisted setting.

Finally, in Section 5 we prove Theorem 1. The proof is by a method of continuity argument for the initial metric, similar to the method of Tian-Zhu in [26]. The main difference between the two approaches is the use of different norms to measure the distance from a metric to a Kähler-Einstein metric. In [26], the distance between two metrics is measured by setting

$$\|g - g'\|_{C^\ell(M)} = \inf_{\Phi} \|g - \Phi^*(g')\|_{C^\ell(M)},$$

where the norm on the right hand side is computed with respect to a fixed metric, and the infimum runs over all diffeomorphisms of  $M$ . In this paper, we measure the distance between an evolving metric and a twisted Kähler-Einstein metric, using instead a  $C^0$ -norm on Kähler potentials. More precisely, fixing a twisted Kähler-Einstein metric  $g_{tKE}$ , we can write  $\tau^*g = g_{tKE} + i\partial\bar{\partial}\phi_\tau$  for any biholomorphism  $\tau$  of  $(M, J)$  which fixes  $\alpha$ . Then, we set

$$d(g) = \inf_{\tau} \text{osc}\phi_\tau.$$

We believe that using this norm makes some of the arguments more transparent. Moreover, working with Kähler potentials allows us to avoid using a result analogous to Chen-Sun's generalized uniqueness theorem [5] similarly to Tian-Zhu [24]. Instead, we only need the extension of Bando-Mabuchi's result [1] to the twisted case, which was given by Berndtsson [2].

Before proceeding, we make a note about conventions. In Section 2 we work in the Riemannian setting. We take this approach, since our results hold in the case of the *real* Ricci flow, twisted by a  $(0, 2)$  tensor satisfying a "contracted Bianchi identity". In particular, all geometric quantities are the *Riemannian* quantities. In all subsequent sections, we work in complex coordinates, with the corresponding complex quantities.

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## 2. THE TWISTED $\mathcal{W}$ -FUNCTIONAL

In this section we introduce the twisted analog of some of Perelman's functionals. These functionals will play a crucial role in our later estimates. For this section only, we will consider the *unnormalized* twisted Kähler-Ricci flow, which is the evolution equation

$$(3) \quad \frac{\partial}{\partial t} \omega = -2(\text{Ric}(\omega) - \alpha).$$

This will allow us to use calculations in the existing literature more readily. Note that if  $\omega(t)$  is a solution of the normalized flow (1), then  $\tilde{\omega}(t) = (1 - 2t)\omega(-\log(1 - 2t))$  is a solution of the unnormalized flow with the same initial condition. In particular in our situation the existence time of the flow (3) is  $t \in [0, \frac{1}{2})$ .

**Definition 2.1.** Let  $(M, g, J)$  be a compact Kähler manifold of complex dimension  $n$ , and let  $\alpha$  be a closed, non-negative  $(1, 1)$ -form. Define the twisted entropy functional  $\mathcal{W} : \mathfrak{Met} \times C^\infty(\mathbb{R}) \times \mathbb{R}_{>0} \rightarrow \mathbb{R}$  by

$$\mathcal{W}(g, f, \tau) := (4\pi\tau)^{-n} \int_M (\tau(R - \text{Tr}_g \alpha + |\nabla f|^2) + (f - 2n)) e^{-f} dm$$

where  $dm = \sqrt{\det g}$  denotes the Riemannian volume form of  $g$ , and all quantities are the real quantities.

**Theorem 2.** Suppose the  $(g(t), f(t), \tau(t)) \in \mathfrak{Met} \times C^\infty(\mathbb{R}) \times \mathbb{R}_{>0}$  solves the coupled system of partial differential equations

$$(4) \quad \frac{\partial}{\partial t} g = -2(\text{Ric}(g) - \alpha)$$

$$(5) \quad \frac{\partial}{\partial t} f = -\Delta_g f + |\nabla f|_g^2 - R(g) + \text{Tr}_g \alpha + \frac{n}{\tau},$$

$$(6) \quad \frac{d}{dt}\tau = -1,$$

on the interval  $[0, T]$ . Then

$$\frac{d}{dt}\mathcal{W}(g(t), f(t), \tau(t)) = 2\tau \int_M \left| Ric(g) + \nabla\nabla f - \alpha - \frac{g}{2\tau} \right|_g^2 (4\pi\tau)^{-n} e^{-f} dm.$$

In particular,  $\mathcal{W}(g(t), f(t), \tau(t))$  is monotonically increasing in  $t$ .

*Proof.* For the proof, we work in *real* coordinates. From now on, we shall suppress the dependence on  $g$ , with the understanding that all Laplacians, curvatures, traces and inner products are computed with respect to  $g$ , unless otherwise noted. Denote by  $\mathcal{W}(t) := \mathcal{W}(g(t), f(t), \tau(t))$ . Following the computation of the variational formula for Perelman's entropy functional [4, 7] we find the variational formula for the twisted entropy functional is given by

$$(7) \quad \begin{aligned} (4\pi\tau)^n \frac{d}{dt}\mathcal{W}(t) &= \int_M -\tau \left\langle \frac{\partial}{\partial t}g, Ric - \alpha + \nabla\nabla f - \frac{1}{2\tau}g \right\rangle e^{-f} dm \\ &+ \int_M \left( \frac{1}{2} \text{Tr} \frac{\partial}{\partial t}g - \frac{\partial}{\partial t}f + \frac{n}{\tau} \right) \left[ \tau(R - \text{Tr}\alpha + 2\Delta f - |\nabla f|^2) \right. \\ &\quad \left. + f - 2n - 1 \right] e^{-f} dm \\ &- \int_M \left( R - \text{Tr}\alpha + |\nabla f|^2 - \frac{n}{\tau} \right) e^{-f} dm \end{aligned}$$

Plugging in the evolution equations (4), (5), (6), we obtain

$$\begin{aligned} (4\pi\tau)^n \frac{d}{dt}\mathcal{W}(t) &= \int_M -2\tau \left\langle Ric - \alpha, Ric - \alpha + \nabla\nabla f - \frac{1}{2\tau}g \right\rangle e^{-f} dm \\ &+ \int_M (\Delta f - |\nabla f|^2)(\tau(R - \text{Tr}\alpha + 2\Delta f - |\nabla f|^2) + f) e^{-f} dm \\ &- \int_M \left( R - \text{Tr}\alpha + |\nabla f|^2 - \frac{n}{\tau} \right) e^{-f} dm \end{aligned}$$

where in the second line we have used that  $\int_M (\Delta f - |\nabla f|^2) e^{-f} dm = 0$ . Since  $\alpha$  is a closed,  $(1, 1)$ -form, it satisfies the “contracted Bianchi identity”

$$(8) \quad \nabla_i \text{Tr}\alpha = 2g^{jp} \nabla_p \alpha_{ij}.$$

Using this identity, the second term can be manipulated as follows;

$$\begin{aligned}
& \int_M (\Delta f - |\nabla f|^2)(\tau(R - \text{Tr}\alpha + 2\Delta f - |\nabla f|^2) + f)e^{-f} dm \\
&= \int_M (\Delta f - |\nabla f|^2)(2\tau\Delta f - \tau|\nabla f|^2)e^{-f} dm \\
&\quad - \int_M |\nabla f|^2 e^{-f} dm - \tau \int_M \langle \nabla f, \nabla(R - \text{Tr}\alpha) \rangle e^{-f} dm \\
&= \tau \int_M \langle -\nabla f, \nabla(2\Delta f - |\nabla f|^2) \rangle e^{-f} dm \\
&\quad - \int_M \Delta f e^{-f} dm - 2\tau \int_M g^{jp} g^{ki} (\nabla_p R_{ij} - \nabla_p \alpha_{ij}) \nabla_k f e^{-f} dm \\
&= -2\tau \int_M g^{ij} \nabla_i f (\nabla_j \Delta f - \langle \nabla f, \nabla_j \nabla f \rangle) e^{-f} dm \\
&\quad + 2\tau \int_M g^{jp} g^{ki} [(R_{ij} - \alpha_{ij}) \nabla_p \nabla_k f - \nabla_p f \nabla_k f (R_{ij} - \alpha_{ij})] e^{-f} dm \\
&\quad + 2\tau \int_M \left\langle \frac{g}{2\tau}, \nabla \nabla f \right\rangle e^{-f} dm \\
&= 2\tau \int_M \left[ \left\langle \nabla \nabla f, Ric - \alpha + \nabla \nabla f - \frac{1}{2\tau} g \right\rangle \right] e^{-f} dm.
\end{aligned}$$

Moreover, the third term can be written as

$$\begin{aligned}
& - \int_M \left( R - \text{Tr}\alpha + |\nabla f|^2 - \frac{n}{\tau} \right) e^{-f} dm \\
&= 2\tau \int_M \left\langle \frac{-1}{2\tau} g, Ric - \alpha + \nabla \nabla f - \frac{1}{2\tau} g \right\rangle e^{-f} dm.
\end{aligned}$$

Combining these three expressions we obtain

$$\frac{d}{dt} \mathcal{W}(t) = 2\tau \int_M \left| Ric - \alpha + \nabla \nabla f - \frac{1}{2\tau} g \right|_g^2 (4\pi\tau)^{-n} e^{-f} dm,$$

which proves the proposition.  $\square$

*Remark.* Note that this computation works for any *real* Ricci flow twisted by any  $(0, 2)$ -tensor  $B$  satisfying the contracted Bianchi identity (8) with respect to all metrics along the flow.

We define the twisted  $\mu$  functional as follows;

**Definition 2.2.** *The functional  $\mu : \mathfrak{Met} \times \mathbb{R}_{>0} \rightarrow \mathbb{R}$  is defined by*

$$\mu(g, \tau) := \inf \{ \mathcal{W}(g, f, \tau) : f \in C^\infty(M, \mathbb{R}) \text{ satisfies } (g, f, \tau) \in \mathcal{X} \}$$

where

$$\mathcal{X} := \left\{ (g, f, \tau) : \int_M (4\pi\tau)^{-n} e^{-f} dm = 1 \right\}$$

Many of the properties of the twisted  $\mu$  functional carry over from the standard Ricci flow. We list them, without proof, in the following proposition. For details, we refer the reader to [7].

**Proposition 2.1.** *The twisted  $\mu$  functional has the following properties;*

- (i) *For any  $c \in \mathbb{R}_{>0}$  we have  $\mu(cg, c\tau) = \mu(g, \tau)$ .*
- (ii) *For a fixed  $\tau \in \mathbb{R}_{>0}$ , the function  $\mu(g, \tau)$  is continuous in the  $C^2$  topology on  $\mathfrak{Met}$ .*
- (iii) *For any  $(g, \tau)$  there exists a function  $f \in C^\infty(M, \mathbb{R}) \cap \mathcal{X}$  such that  $\mathcal{W}(g, f, \tau) = \mu(g, \tau)$ . In particular,  $\mu(g, \tau) > -\infty$ . Moreover,  $f$  satisfies the nonlinear elliptic equation*

$$(9) \quad \tau(2\Delta f - |\nabla f|^2 + R - \text{Tr}_g \alpha) + f - 2n = \mu(g, \tau)$$
- (iv) *Let  $(g, \tau) \in \mathfrak{Met} \times \mathbb{R}_{>0}$  satisfy equations (4) and (6) on  $[0, T]$ . Then functional  $\mu(g, \tau)$  is monotonically increasing. More precisely, for any  $t_0 \in [0, T]$ , if  $f(t_0) \in \mathcal{X}$  is the minimizer whose existence is guaranteed by (iii) we have*

$$\begin{aligned} & \frac{d}{dt} \Big|_{t=t_0} \mu(g, \tau) \\ & \geq 2\tau(t_0) \int_M \left| \text{Ric}(t_0) + \nabla \nabla f(t_0) - \alpha - \frac{g(t_0)}{2\tau(t_0)} \right|^2 (4\pi\tau(t_0))^{-n} e^{-f(t_0)} dm_{t_0} \\ & \text{in the sense of } \liminf \text{ backwards difference quotients. In particular,} \\ & \text{for } r > 0 \text{ we have} \end{aligned}$$

$$\mu(g(t_0), r^2) \geq \mu(g(0), r^2 + t_0)$$

which follows by taking  $\tau(t) = t_0 + r^2 - t$ .

It follows from Proposition 2.1 (i) and (iv), and the relation between the normalized and unnormalized twisted Kähler Ricci flow, that  $\mu(g, \frac{1}{2})$  is monotonically increasing along the normalized flow. For convenience, we record this in the following lemma.

**Lemma 2.1.** *The quantity  $\mu(g, \frac{1}{2})$  is monotonically increasing along the normalized twisted Kähler-Ricci flow. Moreover,  $\mu(g, \frac{1}{2})$  is uniformly bounded above by a topological constant,*

$$\mu(g, \frac{1}{2}) \leq \log [(2\pi)^{-n} \text{Vol}(M)].$$

*Proof.* This follows immediately by substituting

$$f = -n \log(2\pi) + \log(\text{Vol}(M)),$$

into the  $\mathcal{W}$ -functional, using the relation between the real scalar curvature and the complex scalar curvature, and the semi-positivity of  $\alpha$ .  $\square$

In the remainder of this section we prove a non-collapsing estimate along the twisted Kähler-Ricci flow, extending Perelman's estimates for the Kähler-Ricci flow. In the untwisted case, this result is an improvement (due to

Perelman), of Perelman's original non-collapsing result [14, 17]. For our later applications, it will be important to prove *effective* estimates, with clear dependence on the geometry of the initial metric  $g_0$ . First, we need the following easy lemma.

**Lemma 2.2.** *Fix  $x \in M$  and  $t \in [0, \frac{1}{2})$ , and suppose there is an  $r > 0$  such that  $|R(g(t)) - \text{Tr}_{g(t)}\alpha| \leq \frac{K}{r^2}$  on  $B(x, r)$ . Then there exists  $r' \in (0, r]$  such that*

- (i)  $|R - \text{Tr}_g\alpha| \leq \frac{K}{r'^2}$  on  $B(x, r')$
- (ii)  $(r')^{-2n} \text{Vol}(B(x, r')) \leq r^{-2n} \text{Vol}(B(x, r))$
- (iii)  $\text{Vol}(B(x, r')) \leq 3^{2n} \text{Vol}(B(x, r'/2))$ .

*Proof.* The first item holds for any  $r' \leq r$ . By the standard expansion of the volume of a geodesic ball we have

$$\lim_{k \rightarrow \infty} \frac{\text{Vol}(B(x, r/2^k))}{\text{Vol}(B(x, r/2^{k+1}))} = 2^{2n}$$

Hence, there is a  $k < \infty$  such that

$$(10) \quad \frac{\text{Vol}(B(x, r/2^k))}{\text{Vol}(B(x, r/2^{k+1}))} \leq 3^{2n},$$

and if  $l < k$ , then

$$(11) \quad \frac{\text{Vol}(B(x, r/2^l))}{\text{Vol}(B(x, r/2^{l+1}))} > 3^{2n}.$$

Choosing  $r' = r/2^k$ , item (iii) follows from (10), while item (ii) follows by iterating the inequality (11)  $\square$

We first prove the non-collapsing estimate along the *unnormalized* twisted Kähler-Ricci flow. The corresponding estimate along the tKRF is then obtained by rescaling.

**Proposition 2.2.** *Let  $\tilde{g}(s)$  be a solution of the unnormalized twisted Kähler-Ricci flow with  $\tilde{g}(0) = g_0$ . Fix a number  $\rho > 0$ , and define*

$$A(\rho) := \inf_{\tau \in [0, \frac{1}{2} + \rho^2]} \mu(g(0), \tau) > -\infty.$$

*Then, for all  $(x, t) \in M \times [0, \frac{1}{2})$  and  $0 < r \leq \rho$  such that*

$$r^2 |R(\tilde{g}(s)) - \text{Tr}_{\tilde{g}(s)}\alpha| \leq K \text{ on } B_{\tilde{g}(s)}(x, r),$$

*there holds*

$$\text{Vol}_{\tilde{g}(s)}(B(x, r)) \geq \kappa(K, \rho) r^{2n},$$

*where  $\kappa(K, \rho) = \exp(A(\rho) + 2n + n \log(4\pi) - 3^{2n+2} - K)$ .*

*Proof.* For simplicity, we suppress the dependence on  $\tilde{g}(s)$ . Fix a point  $x \in M$ , a radius  $r \in (0, \rho]$  and a time  $t \in [0, 1/2)$ . Suppose that

$$\text{Vol}(B(x, r)) < \kappa r^{2n}.$$

Let  $r' \in (0, r]$  be the number provided by Lemma 2.2. Then we have  $r'^2 |R - \text{Tr} \alpha| \leq K$  on  $B(x, r')$ , and

$$\text{Vol}(B(x, r')) < \kappa r'^{2n}.$$

In order to ease notation, we now set  $r = r'$ . Let  $\phi : [0, \infty) \rightarrow \mathbb{R}$  be the function which is 1 on  $[0, 1/2)$ , decreases linearly to zero on  $(1/2, 1]$ , and is identically zero on  $[1, \infty)$ . Then we set

$$u(y) = e^C \phi(r^{-1}d(x, y))$$

where  $C$  is chosen so that

$$(4\pi)^n = e^{2C} r^{-2n} \int_{B(x, r)} \phi(r^{-1}d(x, y))^2 dm(y).$$

It follows immediately from the definition of  $\phi$  that

$$C > \frac{n}{2} \log(4\pi) - \frac{1}{2} \log(\kappa).$$

We now plug the function  $u$  into the twisted entropy functional to get

$$\begin{aligned} \mathcal{W}(g(t), u, r^2) &\leq 8(4\pi)^{-n} r^{-2n} e^{2C} [\text{Vol}(B(x, r)) - \text{Vol}(B(x, r/2))] \\ &\quad + K - 2n - 2C \\ &\leq 3^{2n+2} (4\pi)^{-n} r^{-2n} e^{2C} \text{Vol}(B(x, r/2)) + K - 2n - 2C \\ &\leq 3^{2n+2} (4\pi)^{-n} r^{-2n} e^{2C} \int_M \phi^2(y) dm(y) + K - 2n - 2C \\ &< 3^{2n+2} + K - 2n - n \log(4\pi) + \log(\kappa). \end{aligned}$$

Since  $A(\rho) \leq \mu(g(0), t + r^2) \leq \mu(g(t), r^2)$  for any  $r \in [0, \rho]$ , we obtain

$$A(\rho) < 3^{2n+2} + K - 2n - n \log(4\pi) + \log(\kappa),$$

from which it follows that  $\kappa > \exp(A(\rho) + 2n + n \log(4\pi) - 3^{2n+2} - K)$ .  $\square$

Along the unnormalized flow we obtain;

**Proposition 2.3.** *Let  $g(t)$  be a solution of the normalized twisted Kähler-Ricci flow. Fix a number  $\rho > 0$ . Then, for all  $(x, t) \in M \times [0, \infty)$  and  $0 < r \leq e^{t/2} \rho$  such that  $r^2 |R(g(t)) - \text{Tr}_{g(t)} \alpha| \leq K$  on  $B(x, r)$  there holds*

$$\text{Vol}_{g(t)}(B(x, r)) \geq \kappa(K, \rho) r^{2n}$$

where  $\kappa(K, \rho)$  is defined in Proposition 2.2.

*Proof.* The proof is just an exercise in scaling. We include the details for completeness. Define a solution of the unnormalized tKRF by setting  $\tilde{g}(s) = (1 - 2s)g(t(s))$  where  $t(s) = -\ln(1 - 2s)$ . Set  $\tilde{r} = e^{-t/2}r$ . Then  $0 < r \leq \rho$ , and we have

$$\tilde{r}^2 |R(\tilde{g}) - \text{Tr}_{\tilde{g}} \alpha| = r^2 |R(g) - \text{Tr}_g \alpha| \leq K$$

on  $B_{\tilde{g}}(x, \tilde{r}) = B_g(x, r)$ . Thus, we can apply the non-collapsing estimate of Proposition 2.2 to  $\tilde{g}(s)$  to obtain

$$\text{Vol}_{\tilde{g}}(B(x, \tilde{r})) \geq \kappa(K, \rho) \tilde{r}^{2n}.$$



Replacing  $\tilde{g}$  with  $g$  proves the proposition.  $\square$

### 3. PERELMAN TYPE ESTIMATES AND THE SOBOLEV INEQUALITY

In this section we extend Perelman's bounds [17] on the scalar curvature and diameter along the Kähler-Ricci flow to the twisted flow. We will need effective estimates in our application, so we are careful to keep track of constants. In addition one difference with the arguments in [17] is that we bound  $u$  independently of the diameter. This may be of independent interest in situations when the diameter is not bounded.

Since the twisted Kähler-Ricci flow (tKRF) preserves the cohomology class of  $\omega$ , we can write it in terms of the Kähler potential  $\phi$  defined by  $\omega(t) = \omega_0 + i\partial\bar{\partial}\phi(t)$ . Let  $u(t)$  be the twisted Ricci potential defined by

$$i\partial\bar{\partial}u = \omega + \alpha - \text{Ric}(\omega).$$

Then on the level of potentials the twisted Kähler-Ricci flow is given by

$$(12) \quad \dot{\phi} = \log \left( \frac{(\omega_0 + i\partial\bar{\partial}\phi)^n}{\omega_0^n} \right) + \phi + u(0),$$

up to the addition of a time dependent constant. At each time we can normalize the twisted Ricci potential  $u$ , so that

$$\int_M e^{-u} dm = \int_M dm = V.$$

This normalization implies that  $u$  must vanish somewhere. We can then normalize the potentials along the flow by setting  $\phi(0) = 0$ , and

$$\frac{\partial}{\partial t}\phi = u,$$

which gives an equation equal to (12), up to adding a time dependent constant.

Differentiating (12), the evolution of  $u$  is given by

$$(13) \quad \frac{\partial}{\partial t}u = \Delta u + u - c(t),$$

where  $c(t)$  is a time-dependent constant. We can compute  $c$  from the normalization condition, since

$$0 = \frac{d}{dt} \int_M e^{-u} dm = \int_M (-\Delta u - u + c + \Delta u) e^{-u} dm,$$

so we need

$$c = \int_M u e^{-u} dm < 0.$$

where the inequality follows immediately from Jensen's inequality.

The following is analogous to the weighted Poincaré inequality of Futaki [10](see also, [25]). The proof is identical, using that  $\alpha$  is a non-negative form.

**Lemma 3.1.** *For any  $f$  on  $M$  we have*

$$\frac{1}{V} \int_M f^2 e^{-u} dm \leq \frac{1}{V} \int_M |\nabla f|^2 e^{-u} dm + \left( \frac{1}{V} \int_M f e^{-u} dm \right)^2.$$

A consequence is the following monotonicity.

**Lemma 3.2.**

$$\frac{d}{dt} c = \frac{d}{dt} \int_M u e^{-u} dm \geq 0.$$

*Proof.* We simply compute

$$\begin{aligned} \frac{d}{dt} \int_M u e^{-u} dm &= \int_M (\Delta u + u - c - u \Delta u - u^2 + cu + u \Delta u) e^{-u} dm \\ &= \int_M (\Delta u - u^2 + cu) e^{-u} dm \\ &= \int_M |\nabla u|^2 e^{-u} dm - \int_M u^2 e^{-u} dm + \frac{1}{V} \left( \int_M u e^{-u} dm \right)^2, \end{aligned}$$

but this last expression is non-negative by the weighted Poincaré inequality applied to  $f = u$ .  $\square$

Before proceeding, we note the following lemma, which follows immediately from the maximum principle.

**Lemma 3.3.** *Along the twisted Kähler-Ricci flow we have*

$$R - \text{Tr}_g \alpha \geq (R - \text{Tr}_g \alpha)^-(0).$$

*In particular,  $\Delta u \leq n + (R - \text{Tr}_g \alpha)^-(0)$ .*

**Lemma 3.4.** *Along the twisted Kähler-Ricci flow we have the lower bound*

$$u(t) > -n + c(0) - \max(R - \text{Tr}_g \alpha)^-(0).$$

*Proof.* The argument of Perelman, as communicated by Sesum-Tian [17] carries over to prove that  $u(t) > -B$  for some constant  $B > 0$ . In order to obtain the effective estimate, we observe that by equation (13) and the monotonicity of  $c(t)$  we have

$$\frac{\partial}{\partial t} u = n - c(t) + \text{Tr}_g \alpha - R + u \leq n - c(0) + \max(R - \text{Tr}_g \alpha)^-(0) + u.$$

Set  $C = n - c(0) + \max(R - \text{Tr}_g \alpha)^-(0)$ , then it follows that, for any  $t_0$

$$u(t) < e^{t-t_0} (u(t_0) + C) - C$$

The lower bound for  $u$  implies that, for all  $t$ , we have

$$u(t) \geq -C = -n + c(0) - \max(R - \text{Tr}_g \alpha)^-(0).$$

$\square$

Next we need to find evolution equations for  $|\nabla u|^2$  and  $\Delta u$ . Standard calculations give the following.

**Lemma 3.5.** *We have*

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla u|^2 &= \Delta|\nabla u|^2 + |\nabla u|^2 - \alpha^{j\bar{k}}\nabla_j u \nabla_{\bar{k}} u - |\nabla\nabla u|^2 - |\nabla\bar{\nabla}u|^2 \\ \frac{\partial}{\partial t}\Delta u &= \Delta(\Delta u) + \Delta u - |\nabla\bar{\nabla}u|^2.\end{aligned}$$

Using these evolution equations the proof of the following lemma is identical to that in [17], again using that  $\alpha$  is non-negative. The effective bounds can be read off directly from the proof.

**Lemma 3.6.** *Let  $B = n - c(0) + \max_M(R - \text{Tr}_g\alpha)^-(0)$ . Then we have*

$$\begin{aligned}|\nabla u|^2 &< 200B(u + 200B) \\ |\Delta u| &< 200B(u + 200B).\end{aligned}$$

Let us assume now that we have a constant  $K$  such that

$$|\Delta u|, |\nabla u|^2 < Ku, \text{ wherever } u > K.$$

We can take  $K = 400B$  with the  $B$  as in the previous lemma. For any two numbers  $a < b$  define the set

$$M(a, b) = \{x \in M \mid a < u(x) < b\}.$$

These sets will be used instead of the geodesic annuli in [17].

**Lemma 3.7.** *There is a constant  $\kappa_1 > 0$  such that if  $a > K$  and  $b > a + 2$  then*

$$\text{Vol}(M(a, b)) > \kappa_1 a^{-n},$$

as long as there is a point  $x$  with  $u(x) = a + 1$ .

*Proof.* On the set  $M(a, a + 2)$  we have  $|\nabla u| < \sqrt{K(a + 2)}$ , so  $M(a, a + 2)$  contains the ball of radius

$$\frac{1}{\sqrt{K(a + 2)}}$$

around the point  $x$ . At the same time, on this ball we have

$$|\Delta u| < K(a + 2),$$

so by non-collapsing estimate in Proposition 2.3 the volume of this ball is at least  $\kappa(n + 1, 1)[K(a + 2)]^{-n}$ . This in turn is at least  $\kappa_1 a^{-n}$  for some other constant  $\kappa_1$ .  $\square$

**Lemma 3.8.** *Let  $0 < \epsilon < 1$  and  $k > \max\{\log_2 \kappa_1^{-1/n}, 2\}$ . Suppose that*

$$\text{Vol}(M(2^k, 2^{10k})) < \epsilon,$$

and  $u(x) > 2^{10k}$  for some  $x$ . Then there exist integers  $k_1, k_2 \in [k, 10k]$  with  $k_2 > k_1 + 4$  such that

$$\begin{aligned}\text{Vol}(M(2^{k_1}, 2^{k_2})) &< \epsilon, \\ \text{Vol}(M(2^{k_1+2}, 2^{k_2-2})) &> 2^{-3n} \text{Vol}(M(2^{k_1}, 2^{k_2})).\end{aligned}$$

*Proof.* The first condition is true for any  $k_1, k_2 \in [k, 10k]$  by hypothesis. We claim that for some integer  $p \in [0, 2k - 1]$  we also have

$$\text{Vol}(M(2^{k+2p}, 2^{9k+2-2p})) < 2^{3n} \text{Vol}(M(2^{k+2p+2}, 2^{9k-2p})),$$

in which case we could choose  $k_1 = k + 2p$  and  $k_2 = 9k + 2 - 2p$ . Otherwise we would have

$$\begin{aligned} 1 > \epsilon > \text{Vol}(M(2^k, 2^{9k+2})) &\geq 2^{3n} \text{Vol}(M(2^{k+2}, 2^{9k})) \geq \dots \\ &\geq 2^{6nk} \text{Vol}(M(2^{5k}, 2^{5k+2})) \geq 2^{6nk} \kappa_1 2^{-5nk} = 2^{nk} \kappa_1. \end{aligned}$$

Since by our assumption  $2^{nk} \kappa_1 > 1$ , we get a contradiction.  $\square$

**Lemma 3.9.** *If  $k_2 > k_1 + 1$ , then we can find  $r_1 \in [2^{k_1}, 2^{k_1+1}]$  and  $r_2 \in [2^{k_2-1}, 2^{k_2}]$  such that*

$$\int_{M(r_1, r_2)} -\Delta u \, dm < CV,$$

for some fixed constant  $C$ , where

$$V = \text{Vol}(M(2^{k_1}, 2^{k_2})).$$

*Proof.* On  $M(2^{k_1}, 2^{k_1+1})$  we have  $|\nabla u|^2 < 2^{k_1+1} K$ , so

$$\int_{M(2^{k_1}, 2^{k_1+1})} |\nabla u|^2 \, dm < 2^{k_1+1} KV.$$

By the coarea formula we have

$$\int_{M(2^{k_1}, 2^{k_1+1})} |\nabla u|^2 \, dm = \int_{2^{k_1}}^{2^{k_1+1}} \int_{\{u=t\}} |\nabla u| \, dS \, dt.$$

Note that by Sard's theorem, the level sets  $\{u = t\}$  are smooth for almost all values of  $t$ , so that the integral on the right hand side makes sense. It follows that there exists  $r_1 \in [2^{k_1}, 2^{k_1+1}]$  such that  $\{u = r_1\}$  is smooth and

$$\int_{\{u=r_1\}} |\nabla u| \, dS < 2KV.$$

Similarly there exists  $r_2 \in [2^{k_2-1}, 2^{k_2}]$  such that  $\{u = r_2\}$  is smooth and

$$\int_{\{u=r_2\}} |\nabla u| \, dS < 2KV.$$

It follows that

$$\begin{aligned} \int_{M(r_1, r_2)} -\Delta u \, dm &\leq \int_{\{u=r_1\}} |\nabla u| \, dS + \int_{\{u=r_2\}} |\nabla u| \, dS \\ &\leq 4KV. \end{aligned}$$

$\square$

**Proposition 3.1.** *There exists a constant  $\epsilon > 0$  such that if for some*

$$k > \max\{\log_2 \kappa_1^{-1/n}, 2\}$$

*we have*

$$\text{Vol}(M(2^k, 2^{10k})) < \epsilon,$$

*then  $u < 2^{10k}$  everywhere.*

*Proof.* Suppose by contradiction that  $u$  takes on larger values, so we can find numbers  $k_1, k_2$  satisfying the conclusions of Lemma 3.8, and then also  $r_1, r_2$  using Lemma 3.9. Let us choose a cutoff function  $\phi$  on  $\mathbf{R}$  such that

$$\phi = \begin{cases} 1 & \text{in } [2^{k_1+2}, 2^{k_2-2}] \\ 0 & \text{outside } [2^{k_1+1}, 2^{k_2-1}]. \end{cases}$$

We can do this in such a way that on the interval  $[2^{k_1+1}, 2^{k_1+2}]$  we have

$$|\phi'(x)| < \frac{2}{2^{k_1+1}} \leq \frac{4}{x},$$

and also on the interval  $[2^{k_2-2}, 2^{k_2-1}]$  we have

$$|\phi'(x)| < \frac{2}{2^{k_2-2}} \leq \frac{4}{x}.$$

So in sum we have  $|\phi'(x)| < 4/x$  for all  $x$ .

Let us now set  $f = A\phi(u)$  for some constant  $A$ , normalised so that

$$\int_M f^2 dm = (2\pi)^n,$$

and we use this as a test function in the  $\mathcal{W}$  functional, with  $\tau = \frac{1}{2}$ . We have

$$\begin{aligned} (2\pi)^n &= \int_M f^2 dm > A^2 \text{Vol}(M(2^{k_1+2}, 2^{k_2-2})) \\ &> A^2 2^{-3n} \text{Vol}(M(2^{k_1}, 2^{k_2})) =: A^2 2^{-3n} V, \end{aligned}$$

so

$$A^2 V < 2^{3n} (2\pi)^n.$$

The monotonicity of  $\mu(g, \frac{1}{2})$  implies that

$$(14) \quad \int_M (-\Delta u) f^2 + 4|\nabla f|^2 - f^2 \ln f^2 - 2n f^2 dm > -C_1$$

for some constant  $C_1$ . We have

$$\int_M (-\Delta u) f^2 dm < A^2 \int_{M(r_1, r_2)} (-\Delta u) dm < A^2 C V < 2^{3n} (2\pi)^n C,$$

with the constant  $C$  from Lemma 3.9. Also

$$|\nabla f|^2 = A^2 |\phi'(u)| |\nabla u| < A^2 \frac{4}{u} \sqrt{K} u < 4A^2 \sqrt{K},$$

assuming  $u > 1$  on  $M(r_1, r_2)$ , ie. if  $r_1 > 1$ . So we get

$$\int_M 4|\nabla f|^2 dm < 16A^2 \sqrt{K} V < 2^{3n+4} (2\pi)^n \sqrt{K}.$$

Also

$$\begin{aligned} \int_M -f^2 \ln f^2 \, dm &= A^2 \int_M -\phi(u)^2 \ln \phi(u)^2 \, dm - \ln A^2 \int_M f^2 \, dm \\ &< A^2 V - 2 \ln A \\ &< 2^{3n} (2\pi)^n - 2 \ln A. \end{aligned}$$

We used that  $-1 < x \ln x \leq 0$  for  $x \in [0, 1]$ . In sum from (14) we get

$$2^{3n} (2\pi)^n C + 2^{3n+4} (2\pi)^n \sqrt{K} + 2^{3n} (2\pi)^n - 2n - 2 \ln A > -C_1.$$

This gives an upper bound  $A < C_2$ . At the same time

$$(2\pi)^n = \int_M f^2 \, dm < A^2 V,$$

so we get  $V > (2\pi)^n A^{-2} > C_2^{-2}$ . Note that  $C_2$  only depends on the entropy of the initial metric of the flow. We can therefore choose  $\epsilon = C_2^{-2}$ .  $\square$

**Corollary 2.** *Along the flow,  $u$  is uniformly bounded.*

*Proof.* Fix a  $k > \max\{\log_2 \kappa_1^{-1/n}, 2\}$ . For any integer  $N > 1$  we have

$$\sum_{i=1}^N \text{Vol}(M(2^{10^{i-1}k}, 2^{10^i k})) < V,$$

where  $V$  is the volume of  $M$ , so if  $N > V/\epsilon$ , then there is an  $i \leq N$  for which

$$\text{Vol}(M(2^{10^{i-1}k}, 2^{10^i k})) < \epsilon.$$

Using the previous Proposition, we must therefore have

$$u < 2^{10^N k}.$$

$\square$

Now, since  $u$  is uniformly bounded above, we obtain uniform upper bounds for  $|\nabla u|$  and  $|\Delta u|$  from Lemma 3.6. We summarize our results in the following

**Theorem 3.** *Suppose that  $g(t)$  evolves along the twisted Kähler-Ricci flow with  $g(0) = g_0$ , then there exists a constant  $C$  depending continuously on the  $C^3$  norm of  $g_0$  (and a uniform lower bound on  $g_0$ ), such that*

$$|u| + |\nabla u|_{g(t)} + |\Delta_{g(t)} u| \leq C.$$

Notice that our estimates *did not* require a diameter bound. In this sense our approach differs from that in [17]. The diameter bound can now be obtained via a simple covering argument.

**Lemma 3.10.** *Along the twisted Kähler-Ricci flow the diameter of  $M$  is uniformly bounded. More precisely,*

$$\text{diam}(M, \omega(t)) \leq \frac{2^{2n} \text{Vol}(M)}{\kappa(K, 1/2)},$$

where  $K$  denotes the uniform upper bound of  $|R - \text{Tr}_g \alpha|$  along the flow, and  $\kappa(K, 1/2)$  is defined in Proposition 2.2

*Proof.* Fix points  $p_1, p_2 \in M$  with  $d(p_1, p_2) = R$ . Let  $\gamma : [0, R] \rightarrow M$  be a length minimizing geodesic connecting  $p_1$  to  $p_2$ . Let  $B_0, B_1, \dots, B_{[R]}$  be balls of radius  $1/2$ , centered at the points  $\gamma(0), \gamma(1), \dots, \gamma([R])$ . These balls are disjoint since  $\gamma$  is length minimizing. By the bound on  $|R - \text{Tr}_g \alpha|$  and Proposition 2.2, we have

$$\text{Vol}(M) \geq \sum_0^{[R]} \text{Vol}(B_i, \omega(t)) \geq R \cdot 2^{-2n} \kappa(K, 1/2).$$

It follows that

$$R \leq \frac{2^{2n} \text{Vol}(M)}{\kappa(K, 1/2)}.$$

□

The above estimates allow us to deduce a uniform Sobolev inequality along the tKRF. Since the proof follows closely the arguments of [11, 27, 28] we list only the key ingredients. The main observation is that the monotonicity of the  $\mu$  functional implies the following uniform, restricted log Sobolev inequality.

**Proposition 3.2.** *Let  $g(t)$  be a solution of the twisted Kähler-Ricci flow defined on  $[0, \infty)$ , with  $g(0) = g_0$ . Define*

$$C_1 = C(n) + 4n \log C_S(M, g(0)) + 4 \frac{\text{Vol}(M)^{-n}}{C_S(M, g_0)^2} + \max_M (R(0) - \text{Tr}_{g_0} \alpha)^-,$$

where  $C_S(M, g_0)$  denotes the Sobolev constant of  $(M, g_0)$ . Then, for all  $\epsilon \in (0, 2]$ ,  $t \in [0, \infty)$  and  $u \in W^{1,2}(M)$  satisfying  $\|u\|_{L^2(g(t))} = 1$ , we have

$$\int_M v^2 \log v^2 dm(t) \leq \epsilon^2 \int_M \left( |\nabla v|^2 + \frac{(R(t) - \text{Tr}_{g(t)} \alpha)}{4} u^2 \right) dm(t) - 2n \log \epsilon + C_1.$$

There is a close relationship between log Sobolev inequalities and estimates for certain heat kernels. Heat kernel estimates are in turn closely related to Sobolev inequalities. In the case of the Ricci flow, the uniform log Sobolev inequality of the above proposition implies a uniform Sobolev inequality along the flow. Of course, a similar estimates hold in the case of the twisted Kähler-Ricci flow. Since the techniques are by now well documented, we omit the details and record only the result. For details in the Kähler case, we refer the reader to [11, 27, 28].

**Proposition 3.3.** *Let  $g(t)$  be a solution of the twisted Kähler-Ricci flow defined on  $[0, \infty)$ , with  $g(0) = g_0$ . Define*

$$C_2 = \sup_M (R(0) - \text{Tr}_{g_0} \alpha)^- + C_1$$

where  $C_1$  is the constant defined in Proposition 3.2. Then there are positive constants  $C(n), \beta(n)$ , depending only on  $n$ , such that for all  $u \in W^{1,2}(M)$  we have

$$\begin{aligned} \left( \int_M u^{2n/(n-1)} dm(t) \right)^{(n-1)/n} &\leq C(n) e^{\beta(n)C_2} \int_M \left( |\nabla u|^2 + \frac{(R - \text{Tr}_{g_0} \alpha)}{4} u^2 \right) dm \\ &+ C(n) e^{\beta(n)C_2} (1 + \max(R(0) - \text{Tr}_{g_0} \alpha)^-) \int_M u^2 dm(t). \end{aligned}$$

#### 4. THE TWISTED MABUCHI ENERGY

In this section we study the tKRF under the assumption that a twisted Kähler-Einstein metric exists. More generally it is enough to assume that the twisted Mabuchi energy is bounded below on the Kähler class  $2\pi c_1(M) - \alpha$ . First, we recall the definition of the twisted Mabuchi energy

**Definition 4.1** ([20] Definition 1.8). *Fix  $\omega_0 \in 2\pi c_1(M) - \alpha$ . For any Kähler form  $\omega_\phi = \omega_0 + i\partial\bar{\partial}\phi$ , we define the twisted Mabuchi energy by its variation at  $\phi$*

$$\delta\mathcal{M}_\alpha(\delta\phi) = - \int_M \delta\phi (R(\omega_\phi) - \text{Tr}_{g_\phi} \alpha - n) \omega_\phi^n dt$$

where  $\delta\phi \in C^\infty(M, \mathbb{R})$ .

The Mabuchi energy of a metric  $\omega$  is obtained by fixing a base point  $\omega_0$  and integrating the variation along a path of metrics connecting  $\omega_0$  to  $\omega$ . It is a basic fact that the result is independent of the path chosen. The connection with twisted Kähler-Einstein metrics is the following.

**Theorem 4.** *Suppose that  $\text{Ric}(\omega) = \omega + \alpha$ . Then the twisted Mabuchi energy is bounded below on the Kähler class  $2\pi c_1(M) - \alpha$ .*

This is a generalization of the result of Bando-Mabuchi [1]. In the twisted case it follows from the results of Chen-Tian [6] (see e.g. Stoppa [20] or Székelyhidi [22]). Along the twisted Kähler-Ricci flow the Mabuchi energy takes on a particularly convenient form.

**Lemma 4.1.** *Suppose that  $\omega(t)$  evolves along the twisted Kähler-Ricci flow with  $\omega(0) = \omega_0$ . Then the Mabuchi energy, with base point  $\omega_0$  is given by*

$$\mathcal{M}_\alpha(\omega_0, \omega(t)) = - \int_0^t \int_M |\nabla u|^2(s) \omega(s)^n ds.$$

*In particular, if the Mabuchi energy is bounded below, then*

$$\lim_{t \rightarrow \infty} \int_M |\nabla u|^2(t) \omega(t)^n = 0.$$

*Proof.* The first assertion follows directly from computation, so we omit the details. For the second assertion, observe that since the Mabuchi energy is bounded below, there exists times  $t_i \in [i, i+1]$  such that

$$\lim_{i \rightarrow \infty} \int_M |\nabla u|^2(t_i) \omega(t_i)^n = 0.$$



This extends to the full sequence by observing that using Theorem 3, we have the differential inequality

$$\frac{\partial}{\partial t} Y(t) \leq CY(t)$$

where  $Y(t) = \int_M |\nabla u|^2(t) \omega(t)^n$  and  $C$  is independent of time. For details in the untwisted case, see Phong-Sturm [15].  $\square$

This estimate allows us to improve the estimates in Theorem 3.

**Proposition 4.1.** *Suppose that the twisted Mabuchi energy is bounded below on  $2\pi c_1(M) - \alpha$ . Then*

$$\lim_{t \rightarrow \infty} |u(t)| + |\nabla u(t)| + |\Delta u(t)| = 0.$$

*Proof.* This is identical to the KRF. See, for instance Phong-Song-Sturm-Weinkove [16].  $\square$

We can now employ these estimate to study the behaviour of the twisted  $\mu$  functional along the tKRF.

**Proposition 4.2.** *Let  $f_t$  be the function achieving  $\mu(g(t), \frac{1}{2})$ , where  $g(t)$  is a solution of the twisted Kahler-Ricci flow with initial value  $g(0) = g_0$ . Then the following estimates hold along the twisted Kähler-Ricci flow.*

- (i) *There exists a constant  $C_1 = C_1(g_0)$  such that  $\sup_M |f_t| \leq C_1$ .*
- (ii) *There exists a subsequence of times  $t_i \in [i, i+1]$  such that*

$$\lim_{i \rightarrow \infty} \left( \int_M |\nabla f_{t_i}|^2 dm_{t_i} \right)^2 + \lim_{i \rightarrow \infty} \int_M |\Delta f_{t_i}|^2 dm_{t_i} = 0.$$

- (iii) *Along the sequence  $t_i$  we have*

$$\lim_{i \rightarrow \infty} \int_M f_{t_i} e^{-f_{t_i}} dm_{t_i} = (2\pi)^n \log((2\pi)^{-n} \text{Vol}(M))$$

*Proof.* The proof of (i) follows [26] closely, so we will only outline the argument. We begin by proving the first bound. For ease of notation, we suppress the dependence on  $t$ . From (9) together with the bounds on  $R - \text{Tr}_g \alpha$  and  $\mu(g, 1/2)$ , the minimizer  $f$  satisfies

$$\Delta f - \frac{1}{2} |\nabla f|^2 < C - f,$$

from which we get

$$\Delta e^{-f/2} = -\frac{1}{2} e^{-f/2} \left( \Delta f - \frac{1}{2} |\nabla f|^2 \right) > -\frac{1}{2} (C - f) e^{-f/2}.$$

Letting  $h = e^{-f/2}$  we then have a constant  $C_\delta$  for any  $\delta > 0$  such that

$$\Delta h \geq -h^{1+\delta} - C_\delta.$$

Moser iteration, together with the normalization  $\int h^2 dm = (2\pi)^n$  implies an upper bound for  $h$ , i.e. a lower bound  $f \geq -C_1$  for  $f$ .

We turn our attention now to the upper bound. Define

$$E_A = \{x \in M : f(x) < A\}.$$

Using the bound on  $u$  from Theorem 3 and the normalization of  $f$ , we have

$$\int_M e^{-f} e^{-u} dm > C^{-1},$$

for some  $C$ . Using the normalization  $\int_M e^{-u} dm = V$  together with the lower bound  $f \geq -C_1$ , this implies that there exists a sufficiently large  $A$ , and a  $\delta > 0$  such that

$$\int_{E_A} e^{-u} dm > \delta.$$

Using the bound on  $u$  again, we have

$$\begin{aligned} \int_M f e^{-u} dm &= \int_{E_A} f e^{-u} dm + \int_{M \setminus E_A} f e^{-u} dm \\ (15) \quad &\leq AC + (V - \delta)^{\frac{1}{2}} \left( \int_M f^2 e^{-u} dm \right)^{\frac{1}{2}}. \end{aligned}$$

It follows that for some  $C_2$  we have

$$(16) \quad \left( \int_M f e^{-u} dm \right)^2 \leq C_2 + (V - \delta/2) \int_M f^2 e^{-u} dm.$$

Multiplying equation (9) by  $e^{-u}$ , integrating, and using the uniform bounds on  $u$ ,  $R - \text{Tr}_g \alpha$  and  $\mu(g)$ , we obtain

$$\begin{aligned} \int_M |\nabla f|^2 e^{-u} dm &\leq \int_M f e^{-u} dm + C_3 \\ &\leq \frac{\delta}{4V} \int_M f^2 e^{-u} dm + C_4, \end{aligned}$$

for some  $C_3, C_4$ . Substituting this into the weighted Poincaré inequality of Lemma 3.1, and using (16) yields

$$\begin{aligned} \int_M f^2 e^{-u} dm &\leq \int_M |\nabla f|^2 e^{-u} dm + \frac{1}{V} \left( \int_M f e^{-u} dm \right)^2 \\ &\leq \frac{\delta}{4V} \int_M f^2 e^{-u} dm + \left( 1 - \frac{\delta}{2V} \right) \int_M f^2 e^{-u} dm + C_5. \end{aligned}$$

Rearranging this, we get an upper bound

$$\int_M f^2 e^{-u} dm \leq C_6$$

where  $C_6$  can be chosen to depend only on  $g(0)$ . By equation (9) we have  $\Delta f \geq -f - C$ , so the upper bound for  $f$  follows from Moser iteration and the  $L^2$  bound.

We now prove the second and third items. We begin by observing that (17)

$$\mu(g(T)) - \mu(g(0)) \geq \int_0^T \left( \int_M |Ric(g) + \nabla \nabla f - \alpha - g|_g^2 (2\pi)^{-n} e^{-f} dm \right) (s) ds$$

To see this, we fix a partition  $P_N = \{0 = t_0 < t_1 < \dots < t_N = T\}$  of  $[0, T]$ , and write

$$\mu(T) - \mu(0) = \sum_{i=1}^N \frac{\mu(t_i) - \mu(t_{i-1})}{t_i - t_{i-1}} (t_i - t_{i-1}).$$

Let  $f_i$  be the smooth function satisfying  $\mathcal{W}(g_i, f_i, \frac{1}{2}) = \mu(t_i)$ , and let  $f_i(t)$  be the solution to the backwards heat equation (5) on  $[t_{i-1}, t_i]$  with  $f_i(t_i) = f_i$ . Then by the mean value theorem we have

$$\begin{aligned} \frac{\mu(t_i) - \mu(t_{i-1})}{t_i - t_{i-1}} &\geq \frac{\mathcal{W}(g_i, f_i, \frac{1}{2}) - \mathcal{W}(g_{i-1}, f_i(t_{i-1}), \frac{1}{2})}{t_i - t_{i-1}} \\ &= (t_i - t_{i-1}) \frac{d}{dt} \Big|_{t=t_i^*} \mathcal{W} \left( g(t), f_i(t), \frac{1}{2} \right), \end{aligned}$$

for some  $t_i^* \in (t_{i-1}, t_i)$ . Using the result of the computation in the proof of Theorem 2 we have

$$\mu(T) - \mu(0) \geq \sum_{i=1}^N (t_i - t_{i-1}) \left( \int_M |Ric(g) + \nabla \nabla f_i - \alpha - g|_g^2 e^{-f_i} dm \right) (t_i^*).$$

Taking the liminf as  $N \rightarrow \infty$  proves the result. Since  $\mu$  is increasing and bounded above, it follows immediately that the bracketed term on the right hand side of (17) goes to zero along a subsequence of times  $t_i \in [i, i+1]$ . In particular, we have

$$\lim_{i \rightarrow \infty} \int_M |\nabla \nabla f_i - \nabla \nabla u(t_i)|^2 e^{-f_i} dm = 0.$$

Now observe that  $|\nabla \nabla f_i - \nabla \nabla u(t_i)|^2 \geq n^{-1} |\Delta(f - u)|^2$ . Applying Proposition 4.1, we obtain

$$\lim_{i \rightarrow \infty} \int_M |\Delta f_i|^2 e^{-f_i} dm = 0.$$

The second item follows, using the upper bound for  $f$ , and the observation

$$\int_M |\nabla f|^2 dm = - \int_M f \Delta f dm \leq C \left( \int_M |\Delta f|^2 dm \right)^{1/2}$$

where  $C$  depends only on the bound for  $f$ .

Finally, we prove the third item. Here our argument differs somewhat from [26]. First, it follows from Jensen's inequality that

$$\int_M f e^{-f} \leq (2\pi)^n \log((2\pi)^{-n} V).$$

Thus, it suffices to prove a lower bound. Set

$$\tilde{f} := f - V^{-1} \int_M (f - u) e^{-u} dm.$$

By the weighted Poincaré inequality of Lemma 3.1, and the choice of normalization we have

$$\begin{aligned} \int_M (\tilde{f} - u)^2 dm &\leq C \int_M (\tilde{f} - u)^2 e^{-u} dm \\ &\leq C \int_M (|\nabla f|^2 + |\nabla u|^2) e^{-u} dm \end{aligned}$$

for a constant  $C$  depending only on  $g_0$ . From this and the upper bound for  $|f|$  we obtain

$$\begin{aligned} \int_M |\tilde{f} - u| e^{-f} dm &\leq C \int_M |\tilde{f} - u| e^{-u} dm \\ &\leq C' \int_M (\tilde{f} - u)^2 e^{-u} dm \\ &\leq C'' \int_M (|\nabla f|^2 + |\nabla u|^2) e^{-u} dm. \end{aligned}$$

In particular, applying item (ii), we have that

$$(18) \quad \lim_{i \rightarrow \infty} \int_M |\tilde{f}_i - u| e^{-f_i} dm = 0.$$

Moreover, by Jensen's inequality we have

$$\begin{aligned} \tilde{f} - f &= V^{-1} \int_M \log(e^{u-f}) e^{-u} dm \\ &\leq \log \left( V^{-1} \int_M e^{-f} dm \right) \\ &= \log((2\pi)^n V^{-1}). \end{aligned}$$

As a result, we have

$$(2\pi)^{-n} \int_M (u - f) e^{-f} dm \leq \log((2\pi)^n V^{-1}) + \int_M (u - \tilde{f}) e^{-f} dm$$

Rearranging this equation gives

$$(2\pi)^{-n} \int_M f e^{-f} \geq -\|u\|_{C^0} + \log((2\pi)^{-n} V) - (2\pi)^{-n} \int_M |\tilde{f} - u| e^{-f} dm.$$

In particular, by Proposition 4.1 and equation (18) we obtain that

$$\lim_{i \rightarrow \infty} \int_M f_i e^{-f_i} \geq (2\pi)^n \log((2\pi)^{-n} V),$$

which finishes the proof.  $\square$

As a result we obtain the following important corollary;

**Corollary 3.** *Suppose that the twisted Mabuchi energy of  $(M, J)$  is bounded below on the Kähler class  $2\pi c_1(M) - \alpha$ . Let  $\omega(t)$  be a solution of the tKRF, with  $\omega(0) = \omega_0 \in 2\pi c_1(M) - \alpha$ . Then*

$$\lim_{t \rightarrow \infty} \mu \left( \omega(t), \frac{1}{2} \right) = \log \left( (2\pi)^{-n} \text{Vol}(M) \right).$$

Moreover, if  $\omega_0 \in 2\pi c_1(M) - \alpha$  satisfies

$$\mu(\omega_0, \frac{1}{2}) = \log \left( (2\pi)^{-n} \text{Vol}(M) \right)$$

then  $\omega_0$  is a twisted Kähler-Einstein metric.

*Proof.* The first statement follows immediately from Proposition 4.2. We prove the second statement. Let  $\omega(t)$  be a solution of the tKRF with  $\omega(0) = \omega_0$ . Then by the monotonicity of  $\mu$ , and Lemma 2.1 we have

$$\log \left( (2\pi)^{-n} \text{Vol}(M) \right) = \mu(\omega_0, \frac{1}{2}) \leq \mu(\omega(t), \frac{1}{2}) \leq \log \left( (2\pi)^{-n} \text{Vol}(M) \right),$$

Thus,  $\mu(t) = \log \left( (2\pi)^{-n} \text{Vol}(M) \right)$  for all  $t$ . Let  $f$  be any minimizer of  $\mathcal{W}(g(0), \cdot, \frac{1}{2})$ . Then by Proposition 2.1 we have that

$$|\text{Ric}(t_0) + \nabla \bar{\nabla} f - \alpha - g(0)|^2 = 0.$$

By the definition of the Ricci potential, we must have

$$\nabla \bar{\nabla} f = \nabla \bar{\nabla} u,$$

and hence  $f = u + c$  for some constant  $c$ . However, we clearly have that  $\hat{f} = -n \log(2\pi) + \log(\text{Vol}(M))$  is a minimizer of  $\mathcal{W}(g(0), \cdot, \frac{1}{2})$ , which follows by direct computation. As a result we have that  $u$  is a constant. It follows from the normalizations that  $u = 0$ , and  $\omega_0$  is twisted Kähler-Einstein.  $\square$

## 5. THE CONVERGENCE OF THE TWISTED KÄHLER-RICCI FLOW

In this section we prove Theorem 1. The argument builds on Tian-Zhu [26], in that the behaviour of Perelman's entropy along the flow is exploited. One important difference is that our distance function  $d(g)$  below measures the oscillation of the Kähler potentials rather than a  $C^{3,\alpha}$  norm of the metric.

Suppose that  $g_{tKE}$  is a twisted Kähler-Einstein metric on  $M$ , satisfying the equation

$$(19) \quad \text{Ric}(g_{tKE}) = g_{tKE} + \alpha,$$

where  $\alpha$  is a non-negative, closed, (1,1)-form. Write  $G \subset \text{Aut}_0(M)$  for the connected component of the group of biholomorphisms preserving  $\alpha$ , i.e.

$$G = \{ \tau \in \text{Aut}_0(M) : \tau^*(\alpha) = \alpha \}.$$

By Berndtsson's generalization [2] of Bando-Mabuchi's uniqueness result [1], every solution of (19) is given by  $\tau^* g_{tKE}$  for some  $\tau \in G$ .

If  $g = g_{tKE} + i\partial\bar{\partial}\phi$ , and  $\tau \in G$ , let us define  $\phi_\tau$  by

$$\tau^*g = g_{tKE} + i\partial\bar{\partial}\phi_\tau,$$

and let

$$d(g) = \inf\{\text{osc } \phi_\tau : \tau \in G\}.$$

Note that  $d(g)$  is independent of the normalization of the Kähler potentials.

Let us write  $u_g$  for the twisted Ricci potential of  $g$ , normalized in any way we like; i.e.

$$i\partial\bar{\partial}u_g = \omega + \alpha - \text{Ric}(g).$$

Note that we can take  $u_{\tau^*g} = \tau^*u_g$  for any  $\tau \in G$ . We will work with  $\text{osc } u_g$ , which is independent of the normalization, and  $\text{osc } u_{\tau^*g} = \text{osc } u_g$ .

The normalized twisted Kähler-Ricci flow (1) is given by

$$\frac{\partial}{\partial t}\phi(t) = u_{g(t)} + c(t),$$

where  $c(t)$  is a time dependent constant depending on our normalizations.

Theorem 3 implies that there is a constant  $K$  depending on  $g(0)$ , such that  $\text{osc } u_{g(t)} < K$  for all  $t$ . Moreover  $K$  can be chosen uniformly as long as  $g(0)$  is bounded in  $C^3$  relative to  $g_{tKE}$ .

We will need the following smoothing result for the twisted Kähler-Ricci flow.

**Theorem 5.** *Suppose that  $\omega(0) = g_{tKE} + i\partial\bar{\partial}\phi(0)$  for some fixed background metric  $g_{tKE}$ , and  $\text{osc } \phi(0), \text{osc } u(0) < K$  for some  $K$ . Then there exist  $s, C > 0$  depending on  $K$  (and  $g_{tKE}$ ), such that at time  $s$  along the twisted Kähler-Ricci flow starting with  $\omega(0)$ , we have*

$$\begin{aligned} C^{-1}g_{tKE} &< g(s) < Cg_{tKE}, \\ \|g(s)\|_{C^3} &< C, \end{aligned}$$

where the  $C^3$  norm is measured using  $g_{tKE}$ .

*Proof.* This follows from Proposition 2.1 in Székelyhidi-Tosatti [23], and is similar to the result of Song-Tian [19] for the Kähler-Ricci flow. One just has to normalize  $\phi$  and  $u_g$  first in order to bound  $\sup |\phi(0)|$  and  $\sup |u(0)|$ .  $\square$

In addition we will use the following result, which follows from the work of Phong-Song-Sturm-Weinkove [16].

**Theorem 6.** *Suppose that along the twisted Kähler-Ricci flow  $g(t)$  we have  $d(g(t)) < K$  for a constant independent of time. Then  $g(t)$  converges to a twisted Kähler-Einstein metric exponentially fast.*

*Proof.* By Perelman's estimate we can assume that also  $\text{osc } u_{g(t)} < K$  for all  $t$ , increasing  $K$  if necessary. Fix a time  $T$ , and let  $\tau \in G$  such that

$$\tau^*g(T) = g_{tKE} + i\partial\bar{\partial}\phi,$$

with  $\text{osc } \phi < K$ . The Ricci potential still satisfies  $\text{osc } u_{\tau^*g(T)} < K$ , so the smoothing property applied to the twisted KR flow starting with  $\tau^*g(T)$  implies that for some  $s, C$  (depending on  $K$ ) we have

$$\begin{aligned} C^{-1}g_{tKE} &< \tau^*g(T+s) < Cg_{tKE} \\ \|\tau^*g(T+s)\|_{C^3(g_{tKE})} &< C. \end{aligned}$$

Since  $T$  was arbitrary, this shows that up to the action of  $G$ , the metrics along the flow are uniformly bounded in  $C^3$ , relative to  $g_{tKE}$ . It follows that a subsequence converges in  $C^{2,\alpha}$ , and the limit is necessarily a twisted Kähler-Einstein metric, since  $u$  tends to a constant by Proposition 4.1. We can also obtain exponential convergence without needing the action of  $G$ , by following the argument of Phong-Song-Sturm-Weinkove [16]. Indeed the uniform  $C^3$  bound implies that the first eigenvalue of  $\bar{\partial}$  on  $TM$  (which is invariant under the action of  $G$ ) is bounded away from zero uniformly.  $\square$

We now show that the distance  $d(g)$  is continuous with respect to the  $C^3$  metric on  $g$ .

**Lemma 5.1.** *If  $g_k \rightarrow g$  in  $C^3$  (with respect to  $g_{tKE}$ ), then  $d(g_k) \rightarrow d(g)$ .*

*Proof.* Recall that for  $\tau \in G$  we wrote

$$\tau^*g = g_{tKE} + i\partial\bar{\partial}\phi_\tau,$$

and

$$d(g) = \inf\{\text{osc } \phi_\tau : \tau \in G\}.$$

We first prove that there exists a  $\tau \in G$  realizing this infimum. Let  $\tau_k \in G$  be a sequence so that  $d(g) = \lim \text{osc } \phi_{\tau_k}$ . We can choose a constant  $K$  such that

$$\begin{aligned} \text{osc } u_g &< K \\ \text{osc } \phi_{\tau_k} &< K. \end{aligned} \tag{20}$$

Using the smoothing property (applied to the twisted KR flow  $\tau_k^*g(t)$ ) we can find  $s, C$  such that

$$C^{-1}g_{tKE} < \tau_k^*g(s) < Cg_{tKE} \tag{21}$$

Fix a point  $p \in M$ . Choosing a subsequence of the  $\tau_k$  we can assume that  $\tau_k(p) \rightarrow q$  for some  $q \in M$ . From (21) it follows that for large enough  $k$ , each  $\tau_k$  maps an open coordinate neighborhood  $B_p$  about  $p$  to a coordinate neighborhood about  $q$ . The component functions of these  $\tau_k$  are then given by uniformly bounded holomorphic functions on  $B_p$ , so after choosing a further subsequence, we can assume that the  $\tau_k$  converge when restricted to the half ball  $\frac{1}{2}B_p$ . The open sets  $\frac{1}{2}B_p$  cover  $M$ , so we can choose a finite subcover, and a subsequence of the  $\tau_k$  will then converge over all of  $M$  to a holomorphic map  $\tau_\infty : M \rightarrow M$ . Taking the limit in (21) we see that  $\tau$  is injective, and it is an open map so it is also surjective. Moreover,  $\tau$  clearly preserves  $\alpha$ , and the connected component of the identity in  $\text{Aut}(M)$  is closed, so  $\tau_\infty \in G$ . It follows that  $d(g) = \text{osc } \phi_{\tau_\infty}$ .

Suppose now that  $g_k \rightarrow g$  in  $C^3$ , and  $\tau \in G$  realizes the infimum  $d(g)$ . Since  $\tau^*g_k \rightarrow \tau^*g$ , using the same  $\tau$  to bound each  $d(g_k)$  we find that

$$\limsup d(g_k) \leq d(g).$$

For the converse inequality suppose that for each  $k$ ,  $\tau_k \in G$  realizes the infimum  $d(g_k)$ . By the same argument as above (we can choose a uniform  $K$  in (20) for all the  $g_k$ ), up to choosing a subsequence we can assume that  $\tau_k \rightarrow \tau_\infty$ . Then  $\tau_k^*g_k \rightarrow \tau_\infty^*g$ , and using  $\tau_\infty$  to bound  $d(g)$  we get

$$d(g) \leq \liminf d(g_k).$$

This shows that  $d(g) = \lim d(g_k)$ .  $\square$

We will now write  $\mu(g) = \mu(g, \frac{1}{2})$  for Perelman's entropy. We collect here a few of the previous results about  $\mu$ , from Lemma 2.1 and Corollary 3:

- (1)  $\mu$  is continuous in the  $C^3$ -norm, measured relative to  $g_{tKE}$ .
- (2) For any initial metric  $g(0)$ ,  $\mu(g(t))$  is monotonically increasing along the twisted Kähler-Ricci flow  $g(t)$ , and  $\lim \mu(g(t)) = \Lambda$ , where we fix  $\Lambda = \log((2\pi)^{-n} \text{Vol}(M))$ . In particular,  $\Lambda$  is independent of  $g(0)$ .
- (3)  $\mu(g) = \Lambda$  if and only if  $g$  is a twisted KE metric. By Berndtsson's uniqueness theorem this is equivalent to:  $\mu(g) = \Lambda$  if and only if  $g = \tau^*g_{tKE}$  for a biholomorphism  $\tau \in G$  of  $(M, J)$  fixing  $\alpha$ .

The following is a consequence of the smoothing result.

**Lemma 5.2.** *Fix  $K > 0$ , and suppose that*

$$\begin{aligned} 1 &\leq d(g) < K \\ \text{osc } u_{g(t)} &< K, \end{aligned}$$

*for all  $t$  along the twisted KR flow  $g(t)$  with initial metric  $g$ . There exist  $s, C > 0$  depending on  $K$ , and a  $\tau \in G$  such that*

$$\begin{aligned} d(g(s)) &\geq \frac{1}{2}, \\ C^{-1}g_{tKE} &< \tau^*g(s) < Cg_{tKE}, \\ \|\tau^*g(s)\|_{C^3} &< C, \end{aligned}$$

*where the  $C^3$ -norm is measured with respect to the fixed metric  $g_{tKE}$ .*

*Proof.* For any  $\tau \in G$ , the flow  $\tau^*g(t)$  is a solution of the twisted KR flow, and it follows by our assumption that  $\text{osc } u_{\tau^*g(t)} < K$  along the flow with initial metric  $\tau^*g$ . Write

$$\tau^*g(t) = g_{tKE} + i\partial\bar{\partial}\phi(t),$$

so that  $\text{osc } \phi(0) \geq 1$  by the assumption that  $d(g) \geq 1$ . Along the twisted KR flow,

$$\dot{\phi}(t) = u_{\tau^*g(t)} + c(t),$$



where  $c(t)$  is a time dependent constant. So if  $s < (4K)^{-1}$  we have

$$\begin{aligned} \sup \phi(s) &> \sup \phi(0) - Ks + A > \sup \phi(0) - \frac{1}{4} + A \\ \inf \phi(s) &< \inf \phi(0) + Ks + A < \inf \phi(0) + \frac{1}{4} + A, \end{aligned}$$

for some constant  $A$  (the integral of  $c(t)$ ), and so

$$\text{osc } \phi(s) > \text{osc } \phi(0) - \frac{1}{2} \geq \frac{1}{2}.$$

Since this is true for any  $\tau \in G$ , by taking infimum we have

$$d(g(s)) \geq \frac{1}{2}.$$

For the smoothing result, we first choose a  $\tau \in G$  such that

$$\tau^*g = g_{tKE} + i\partial\bar{\partial}\phi,$$

with  $\text{osc } \phi < K$ . Then we can use the smoothing theorem, applied to the flow  $\tau^*g(t)$  with initial metric  $\tau^*g$ , and choose  $s$  even smaller than we did in the previous step if necessary.  $\square$

**Proposition 5.1.** *Fix  $K > 0$ . There exists a  $c > 0$  depending on  $K$ , such that if*

$$\begin{aligned} \mu(g) &> \Lambda - c, \\ d(g), \text{osc } u_{g(t)} &< K, \end{aligned}$$

for all time  $t$  along the twisted KR flow  $g(t)$ , then  $d(g) < 1$ .

*Proof.* We argue by contradiction. Suppose there is a  $K > 0$  for which there is no suitable  $c$ . This means that we can choose a sequence  $g^k$  for which

$$(22) \quad \mu(g^k) > \Lambda - 1/k,$$

and  $d(g^k), \text{osc } u_{g^k(t)} < K$  for all  $t$ , but  $d(g^k) \geq 1$ .

We apply Lemma 5.2. We get  $s, C > 0$ , and a  $\tau_k \in G$  such that

$$\begin{aligned} (23) \quad d(g^k(s)) &\geq \frac{1}{2} \\ C^{-1}g_{tKE} &< \tau_k^*g^k(s) < Cg_{tKE}, \\ \|\tau_k^*g^k(s)\|_{C^3(g_{tKE})} &< C, \\ \mu(\tau_k^*g^k(s)) &= \mu(g^k(s)) \geq \mu(g^k) > \Lambda - 1/k, \end{aligned}$$

where we also used the monotonicity of  $\mu$  along the twisted KR flow. We can choose a subsequence of the  $\tau_k^*g^k(s)$  converging to some  $g_\infty$  in  $C^{2,\alpha}$  and in particular  $\mu(g_\infty) = \Lambda$ , so  $g_\infty = \tau^*g_{tKE}$  for some  $\tau \in G$ . The metric  $g_\infty$  is uniformly equivalent to  $g_{tKE}$  (we can use the constant  $C$  given by the bounds (23)) and so

$$\begin{aligned} \|(\tau^{-1})^*\tau_k^*g^k(s) - g_{tKE}\|_{C^{2,\alpha}(g_{tKE})} &= \|(\tau^{-1})^*\tau_k^*g^k(s) - (\tau^{-1})^*g_\infty\|_{C^{2,\alpha}(g_{tKE})} \\ &= \|\tau_k^*g^k(s) - g_\infty\|_{C^{2,\alpha}(\tau^*g_{tKE})} \rightarrow 0. \end{aligned}$$

Writing

$$(\tau^{-1})^* \tau_k^* g^k(s) = g_{tKE} + i\partial\bar{\partial}\phi_k(s)$$

with  $\phi_k(s)$  normalized to have zero mean, we obtain

$$\phi_k(s) \rightarrow 0 \text{ in } C^{4,\alpha}(g_{tKE}),$$

which contradicts

$$\text{osc}(\phi_k(s)) \geq d(g^k(s)) \geq \frac{1}{2}.$$

□

Next we prove a stability result for the twisted Kähler-Ricci flow. In the untwisted case, a similar result was proved by Sun-Wang [21], using different techniques.

**Proposition 5.2.** *There is a  $\delta > 0$  such that if  $\|g - g_{tKE}\|_{C^3(g_{tKE})} < \delta$ , then the twisted Kähler-Ricci flow starting with  $g$  converges to a twisted Kähler-Einstein metric.*

*Proof.* We can choose a constant  $K > 1$  such that if  $\|g - g_{tKE}\|_{C^3(g_{tKE})} < \frac{1}{2}$ , then  $d(g) < K$ , and also by Perelman's estimate  $\text{osc } u_{g(t)} < K$  for all  $t$ . Let  $c = c(K)$  be the constant given by Proposition 5.1. We can now choose  $\delta < 1$  sufficiently small so that

$$\|g - g_{tKE}\|_{C^3(g_{tKE})} < \delta$$

implies that  $\mu(g) > \Lambda - c$ . By the monotonicity we have

$$\mu(g(t)) > \Lambda - c$$

for all  $t > 0$ . The previous Proposition implies that  $d(g) < 1$  and then also  $d(g(t)) < 1$  for all  $t > 0$  (since at any time if  $d(g(t)) = 1$ , then Proposition 5.1 says  $d(g(t)) < 1$ ). Theorem 6 implies that  $g(t)$  converges to a twisted KE metric. □

Let  $S$  be the set of  $C^5$  metrics  $g$  in the class  $[g_{tKE}]$  such that the twisted KR flow starting with  $g$  converges to a twisted KE metric in  $C^5(g_{tKE})$ . Our goal is to prove that  $S$  is both open and closed. We claim that this implies our main theorem. To see this, observe that the set of  $C^5$  metrics in the class  $[g_{tKE}]$  is convex, and hence connected. It follows that either  $S$  contains all metrics, or  $S$  is empty. But  $g_{tKE} \in S$ , and hence Theorem 1 follows.

**Proposition 5.3.**  *$S$  is open in the  $C^5$  topology.*

*Proof.* Suppose that  $g \in S$ . Then for sufficiently large  $T$  there exists a  $\tau \in G$  such that  $\|\tau^*g(T) - g_{tKE}\|_{C^3(g_{tKE})} < \delta/2$  with the  $\delta$  from the previous result. For a finite time  $t \in [0, T]$  the solution of the twisted KR flow depends smoothly on the initial data, so, since  $\tau \in G$  is fixed, we can choose  $c$  small so that if  $\|h - g\|_{C^5(g_{tKE})} < c$ , then  $\|\tau^*h(T) - \tau^*g(T)\|_{C^3(g_{tKE})} < \delta/2$ . Then

$$\|\tau^*h(T) - g_{tKE}\|_{C^3(g_{tKE})} < \delta,$$

so the stability result implies that the flow starting with  $\tau^*h(T)$ , and hence also the flow starting with  $h$ , converges to a twisted Kähler-Einstein metric.  $\square$

**Proposition 5.4.**  *$S$  is closed in  $C^5$ .*

*Proof.* Suppose that  $g_k \in S$ , and

$$g_k \rightarrow g \text{ in } C^5.$$

By Perelman's estimate, we can choose a  $K > 2$  such that  $\text{osc } u_{g_k(t)} < K$  for all  $k, t$  (using that the  $g_k$  are in a bounded set of metrics in  $C^5$ ). Let  $c$  be the constant given by Proposition 5.1 corresponding to  $K$ . By the properties of  $\mu$ , there exists a  $T$  such that

$$\mu(g(T)) > \Lambda - c.$$

Since the tKRF is stable for finite time, and  $\mu$  is continuous for a  $C^3$  family of metrics, there exists an  $N$  such that

$$\mu(g_k(T)) > \Lambda - c$$

for all  $k > N$ . By monotonicity, it follows that

$$\mu(g_k(t)) > \Lambda - c$$

for all  $k > N$  and  $t \geq T$ . Since we know that  $g_k(t)$  converges to a twisted KE metric  $\tau_k^* g_{tKE}$  for some  $\tau_k \in G$ , it follows that  $(\tau_k^{-1})^* g_k(t)$  converges to  $g_{tKE}$ . Now, since  $(\tau_k^{-1})^* g_k(t)$  converges to  $g_{tKE}$  we can apply Lemma 5.1 to obtain that  $d((\tau_k^{-1})^* g_k(t)) = d(g_k(t))$  converges to  $d(g_{tKE}) = 0$ . In particular, there must be a first time  $t_k \geq T$  for which

$$d(g_k(t_k)) \leq K/2.$$

By Proposition 5.1, at this time we have  $d(g_k(t_k)) < 1$ , so since  $K > 2$  and  $t_k \geq T$  was chosen to be minimal,  $t_k = T$ . But then as before, we have  $d(g_k(t)) < 1$  for all  $t > T$  and  $k > N$ . For any fixed  $t$  we have  $g_k(t) \rightarrow g(t)$  in  $C^3(g_{tKE})$  as  $k \rightarrow \infty$ , and so apply Lemma 5.1 again, we obtain

$$d(g(t)) \leq 1$$

for all  $t \geq T$ . Theorem 6 then implies that  $g(t)$  converges to a twisted KE metric.  $\square$

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